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Policy iteration for american options: overview*

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Abstract — This paper is an overview of recent results by Kolodko and Schoenmakers (2006), Bender and Schoenmakers (2006) on the evaluation of options with early exercise opportunities via policy improvement. Stability is discussed and simulation results based on plain Monte Carlo estimators for conditional expectations are presented.

1 Introduction

The evaluation of American style derivatives on a high dimensional underlying is an important and challenging problem. Typically these derivatives cannot be priced by the classical PDE methods, as the computational cost rapidly increases with the dimension of the underlying. This problem is known as the 'curse of dimensionality'. Only in recent years several approaches have been proposed to overcome this problem. These methods basically rely on Monte Carlo simulation and can be roughly divided into three groups. The first group directly employs a recursive scheme for solving the stopping problem, known as backward dynamic programming. Different techniques are applied to approximate the nested conditional expectations. The stochastic mesh method by Broadie et al. (2000) and the least square regression method of Longstaff and Schwartz

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(2001) are among the most popular approaches in this group. An alternative to backward dynamic programming is to approximate the exercise boundary by simulation, see e.g. Andersen (1999), Ibáñez and Zapatero (2004). The third group relies on a dual approach developed in Rogers (2002), Haugh and Kogan (2004), and in a multiplicative setting by Jamshidian (1997). For a numerical treatment of this approach, see Kolodko and Schoenmakers (2004). By duality, tight price upper bounds may be constructed from given approximative processes.

In this paper we survey a new policy iteration for discretized American options which was recently introduced in Kolodko and Schoenmakers (2006) and Bender and Schoenmakers (2006). The method is mending one of main drawbacks of backward dynamic programming: Suppose exercise can take place at one out of k time instances. Then, in order to obtain the value of the optimal stopping problem via backward dynamic programming, one has to calculate nested conditional expectations of order k. No approximation of the time 0 value is available prior to the evaluation of the kth nested conditional expectations. This prevents the use of plain Monte Carlo simulations for approximating the conditional expectations and requires more complicated approximation procedures for these quantities. For instance, to employ the procedure of Longstaff and Schwartz (2001), one has to choose the number of basis functions and the basis functions themselves, i.e. the approximation procedure must be differently tailored to different derivatives. Contrary, our policy iteration yields approximations of the time 0 value of the value function for every iteration step, which monotonically increase to the Snell envelope. This allows to calculate some approximations of the Snell envelope by plain Monte Carlo simulations. The algorithm converges in the same number of steps as backward dynamic programming does. So theoretically, the algorithm is as good as backward dynamic programming.

After recalling the optimal stopping problem in section 2, we introduce our policy iteration in section 3.1. Note, the policy iteration is different from Howard (1960) policy iteration for backward dynamic programming and can be shown to yield better approximations. Stability of the policy improvement is discussed in section 3.2. It turns out, that the shortfall of the perturbed policy improvement under the theoretical policy improvement converges to zero. Surprisingly, the distance need not convergence, so that the perturbed improvement can even perform better than the theoretical. Section 4 is devoted to simulations. We evaluate the price of basket-put and maximum-call on five assets, which has become a benchmark problem in recent years. The examples show that tight approximations of the option prices can be achieved with a plain Monte Carlo simulation.

2 Optimal stopping in discrete time

It is well known that by the no arbitrage principle the pricing of American options is equivalent to the optimal stopping problem of the discounted derivative under a pricing measure. We now recall some facts about the optimal stopping problem in discrete time.

Suppose $(Z(i)\colon\ i = 0, 1, \ldots, k)$ is a nonnegative stochastic process in discrete time on a probability space $(\Omega, \mathcal{F}, P)$ adapted to some filtration $(\mathcal{F}_i : 0 \le i \le k)$ which satisfies

$$\sum_{i=1}^{k} E|Z(i)| < \infty.$$

We may think of the process Z as a cashflow, which an investor may exercise once. The investors' problem is to maximize his expected gain by choosing the optimal time for exercising. This problem is known as optimal stopping in discrete time.

To formalize the stopping problem we define $\mathcal{S}_i$ as the set of $\mathcal{F}_i$ stopping times taking values in $\{i, \ldots, k\}$. The stopping problem can now be stated as follows:

Find stopping times $\tau^*(i) \in \mathcal{S}_i$ such that for $0 \le i \le k$

$$E^{\mathcal{F}_i}\left[Z(\tau^*(i))\right] = \operatorname{esssup}_{\tau \in \mathcal{S}_i} E^{\mathcal{F}_i}\left[Z(\tau)\right]. \tag{1}$$

The process on the right hand side is called the *Snell envelope* of Z and we denote it by $Y^*(i)$.

We collect some facts, which can be found in Neveu (1975) for example.

1. The Snell envelope Y^* of Z is the smallest supermartingale that dominates Z. It can be constructed recursively by backward dynamic programming:

$$\begin{aligned} Y^*(k) &= Z(k) \\ Y^*(i) &= \max\{Z(i),\ E^{\mathcal{F}_i}[Y^*(i+1)]\}. \end{aligned}$$

2. A family of optimal stopping times is given by

$$\widetilde{\tau}^*(i) = \inf\{i \le j \le k :\ Z(j) \ge Y^*(j)\}.$$

If several optimal stopping families exist, then the above family is the family of first optimal stopping times. In that case

$$\widehat{\tau}^*(i) = \inf\{i \le j \le k : \ Z(j) > Y^*(j)\}$$

is the family of last optimal stopping times.

3 The policy iteration

3.1 Definition of the improvement procedure

Suppose the buyer of the option chooses ad hoc a family of stopping times $(\tau(i) : \ 0 \le i \le k)$ taking values in the set $\{0, \dots, k\}$. We interpret $\tau(i)$ as the time, at which the buyer will exercise his option, provided he has not exercised prior to time i. This interpretation requires the following consistency condition:

Definition 3.1 *A family of integer-valued stopping times $(\tau(i) : \ 0 \le i \le k)$ is said to be* consistent*, if*

$$i \le \tau(i) \le k, \ \tau(k) \equiv k,$$
$$\tau(i) > i \Rightarrow \tau(i) = \tau(i+1), \qquad 0 \le i < k.$$

Indeed, suppose $\tau(i) > i$, i.e. according to our interpretation the investor has not exercised the first right prior to time $i+1$. Then he has not exercised the first right prior to time i, either. This means he will exercise the first right at times $\tau(i)$ and $\tau(i+1)$, which requires $\tau(i) = \tau(i+1)$. A typical example of a consistent stopping family can be obtained by comparison with the still-alive European options

$$\tau(i) := \inf\left\{ j : \ i \le j \le k, \ Z(j) \ge \max_{j+1 \le p \le k} E^{\mathcal{F}_j}\left[Z(p)\right] \right\}. \tag{2}$$

Given some consistent stopping family τ we define a new stopping family by

$$\widetilde{\tau}(i) := \inf\left\{ j : \ i \le j \le k, \ Z(j) \ge \max_{j+1 \le p \le k} E^{\mathcal{F}_j}\left[Z(\tau(p))\right] \right\}. \tag{3}$$

Note, the stopping family $\widetilde{\tau}$ is consistent. In particular $\widetilde{\tau}(k) = k$, since $\max \emptyset = -\infty$. We call $\widetilde{\tau}$ a *one-step improvement* of τ for the following reason: denote by $Y(i;\tau)$ the value process corresponding to the stopping family τ, namely

$$Y(i;\tau) = E^{\mathcal{F}_i}\left[Z(\tau(i))\right].$$

Then the one-step improvement yields a higher value than the given family,

$$Y(i;\widetilde{\tau}) \geq Y(i;\tau).$$

This will be proved in theorem 3.2 below. We note that, for example, the stopping family based on the maximum of still alive Europeans in (2) is the one-step improvement of the trivial stopping family $\tau(i) = i$.

It is natural to iterate this policy improvement: suppose τ_0 is some consistent stopping family. Define, recursively,

$$\begin{aligned} \tau_m &= \widetilde{\tau}_{m-1} \\ Y_m(i) &= Y(i;\tau_m). \end{aligned}$$

It can be shown that $Y_m(i)$ coincides with the time i value of the Snell envelope when $m \geq k - i$. This means the policy improvement algorithm is theoretically as good as backward dynamic programming, but admits to calculate increasing approximations of the Snell envelope at every iteration step.

Remark 3.1 *Given a consistent stopping family τ, $Y(0;\tau)$ is always a lower bound of $Y^*(0)$. From this lower bound an upper bound can be constructed by a duality method developed by Rogers (2002) and Haugh and Kogan (2004). Define,*

$$Y_{up}(\tau) = E\left[\max_{0\leq j\leq k}(Z(j) - M(j))\right], \tag{4}$$

where $M(0) = 0$ and, for $1 \leq i \leq k$,

$$M(i) = \sum_{p=1}^{i}\left(Y(p;\tau) - E^{\mathcal{F}_{p-1}}\left[Y(p;\tau)\right]\right).$$

Remark 3.2 *When τ^* is some optimal stopping family, the supermartingale property of the Snell envelope yields,*

$$\max_{i+1\leq p\leq k} E^{\mathcal{F}_i}\left[Z(\tau^*(p))\right] = E^{\mathcal{F}_i}\left[Y^*(i+1)\right].$$

Thus, the one-step improvement of τ^* is the family of first optimal stopping times. This shows, the latter family is the only fixed point of the one-step improvement.

3.2 Stability

In practice, we cannot expect to know analytical expressions of the conditional expectations on the right hand side of the exercise criterion in (3), but can only calculate approximations. Therefore, a stability result is called for.

Given a consistent stopping family τ and a sequence of $\mathcal{F}_i$-adapted processes $\epsilon^{(N)}(i)$ define

$$\widetilde{\tau}^{(N)}(i) := \inf\left\{ j:\ i \le j \le k,\ Z(j) \ge \max_{j+1 \le p \le k} E^{\mathcal{F}_j}\left[Z(\tau(p))\right] + \epsilon^{(N)}(j) \right\}.$$

The sequence $\epsilon^{(N)}$ accounts for the errors when approximating the conditional expectation. We suppose throughout this section that

$$\lim_{N\to\infty} \epsilon^{(N)}(i) = 0, \quad P\text{-}a.s.$$

We will first show by some simple examples that we must neither expect

$$\widetilde{\tau}^{(N)}(i) \to \widetilde{\tau}(i) \quad \text{in probability}$$

nor

$$Y(0; \widetilde{\tau}^{(N)}) \to Y(0; \widetilde{\tau}).$$

Example (i) Suppose ξ_N is a sequence of independent binary trials with $P(\xi_N = 1) = P(\xi_N = 0) = 1/2$. We define the process $(Z(i):\ i = 0, 1)$ by $Z(0) = Z(1) \equiv 1$. The σ-field $\mathcal{F}_0 = \mathcal{F}_1$ is the one generated by the sequence of trials. Moreover, the sequence of perturbations is defined by $\epsilon^{(N)}(0) = \xi_N/N$ and $\epsilon^{(N)}(1) = 0$. Then, starting with any consistent stopping family τ, we get

$$\widetilde{\tau}^{(N)}(0) = \xi_N.$$

In particular, no subsequence of $\widetilde{\tau}^{(N)}(0)$ converges in probability.

(ii) Let $\Omega = \{\omega_0, \omega_1\}$, $\mathcal{F}$ the powerset of Ω and $P(\{\omega_1\}) = 1/4 = 1 - P(\{\omega_0\})$. We define the process $(Z(i):\ i = 0, 1, 2)$ by $Z(0) = Z(2) = 2$ and $Z(1, \omega_0) = 1$, $Z(1, \omega_1) = 3$. $\mathcal{F}_i$ is the filtration generated by Z. We start with the stopping family $\tau(i) = i$. As $E[Z(1)] = 3/2$, we have

$$Z(0) = 2 \ge \max\{3/2,\ 2\} = \max\{E[Z(1)],\ E[Z(2)]\} = \widehat{Y}(0, \tau).$$

Therefore,

$$\widetilde{\tau}(0) = 0$$

and

$$Y(0;\widetilde{\tau}) = 2.$$

The perturbation sequence $\epsilon^{(N)}$ is defined to be $\epsilon^{(N)}(1) = \epsilon^{(N)}(2) \equiv 0$ and $\epsilon^{(N)}(0) = 1/N$. A straightforward calculation shows, for $N \geq 2$,

$$\widetilde{\tau}^{(N)}(0,\omega_0) = 2, \quad \widetilde{\tau}^{(N)}(0,\omega_1) = 1.$$

Thus,

$$Y(0;\widetilde{\tau}^{(N)}) = 9/4 > 2 = Y(0;\widetilde{\tau}),$$

which is the claimed violation of stability.

The example paints a rather sceptical picture of the stability of the one-step-improvement. Indeed, the best we can now hope for, is

(ia) there is a sequence $\bar{\tau}^{(N)}$ of stopping families such that

$$|\widetilde{\tau}^{(N)}(i) - \bar{\tau}^{(N)}(i)| \to 0 \quad P\text{-}a.s.$$

and, for all N, $\bar{\tau}^{(N)}$ is at least as good as $\widetilde{\tau}$, i.e.

$$Y(i;\bar{\tau}^{(N)}) \geq Y(i;\widetilde{\tau}).$$

(iia) The shortfall of $Y(i;\widetilde{\tau}^{(N)})$ below $Y(i;\widetilde{\tau})$ converges to zero P-a.s.

Note, however, that the convergence of the shortfall as in (iia) is the relevant question, not of the distance as in example (ii), page 352: the shortfall corresponds to a change for the worse of $\widetilde{\tau}^{(N)}$ compared to $\widetilde{\tau}$. As we are interested in an improvement it suffices to guarantee that such a change for the worse converges to zero. An additional improvement of $\widetilde{\tau}^{(N)}$ compared to $\widetilde{\tau}$ due to the error processes $\epsilon^{(N)}$ may be seen as a welcome side effect.

In the remainder of this section we sketch the proof of (ia) and (iia).

Theorem 3.1 *The one-step improvement is stable in the sense of (ia) and (iia).*

Remark 3.3 *It clearly suffices to prove (ia). Indeed,*

$$\left(Y(i;\widetilde{\tau}^{(N)}) - Y(i;\widetilde{\tau})\right)_- \leq \left(Y(i;\widetilde{\tau}^{(N)}) - Y(i;\bar{\tau}^{(N)})\right)_- + \left(Y(i;\bar{\tau}^{(N)}) - Y(i;\widetilde{\tau})\right)_- .$$

By (ia) the second term vanishes and the first converges to zero due to dominated convergence.

In order to construct an appropriate family $\bar{\tau}^{(N)}$ we first derive a criterion for a consistent stopping family $\bar{\tau}$ to be at least as good as $\widetilde{\tau}$. To this end define,

$$\widehat{\tau}(i) := \inf\left\{ j :\ i \le j \le k,\ Z(j) > \max_{j+1 \le p \le k} E^{\mathcal{F}_j}\left[Z(\tau(p))\right] \right\}.$$

Obviously,

$$\widehat{\tau}(i) \ge \widetilde{\tau}(i).$$

Theorem 3.2 *Suppose τ, $\bar{\tau}$ are consistent stopping families and*

$$\widetilde{\tau}(i) \le \bar{\tau}(i) \le \widehat{\tau}(i). \tag{5}$$

Then,

$$Y(i;\bar{\tau}) \ge Y(i;\widetilde{\tau}) \ge \max\left\{ Z(i),\ \max_{p \ge i} E^{\mathcal{F}_i}\left[Z(\tau(p))\right] \right\} \ge Y(i;\tau).$$

Proof. The last inequality is trivial, since $Y(i;\tau) = E^{\mathcal{F}_i}\left[Z(\tau(i))\right]$. To prove the other inequalities we begin with a preliminary consideration. Define

$$\begin{aligned}
\widetilde{Y}(i;\tau) &= \max_{p \ge i} E^{\mathcal{F}_i}\left[Z(\tau(p))\right] \\
\widehat{Y}(i;\tau) &= \max_{p \ge i+1} E^{\mathcal{F}_i}\left[Z(\tau(p))\right].
\end{aligned}$$

Then,

$$\widetilde{Y}(i;\tau) = \mathbf{1}_{\{\tau(i) > i\}} \widehat{Y}(i;\tau) + \mathbf{1}_{\{\tau(i) = i\}} \max\left\{ \widehat{Y}(i;\tau),\ Z(i) \right\}, \tag{6}$$

since, by the consistency of τ,

$$\begin{aligned}
E^{\mathcal{F}_i}\left[Z(\tau(i))\right] &= E^{\mathcal{F}_i}\left[\mathbf{1}_{\{\tau(i)=i\}} Z(i)\right] + E^{\mathcal{F}_i}\left[\mathbf{1}_{\{\tau(i)>i\}} Z(\tau(i+1))\right] \\
&= \mathbf{1}_{\{\tau(i)=i\}} Z(i) + \mathbf{1}_{\{\tau(i)>i\}} E^{\mathcal{F}_i}\left[Z(\tau(i+1))\right].
\end{aligned}$$

Step 1:

$$Y(i;\bar{\tau}) \ge \max\left\{ Z(i),\ \max_{p \ge i} E^{\mathcal{F}_i}\left[Z(\tau(p))\right] \right\} \tag{7}$$

by backward induction over i. The induction base is obvious, since $\tau(k) = \bar{\tau}(k) = k$. Suppose now $0 \le i \le k-1$, and that the assertion is already proved for $i+1$. Note, $\{\bar{\tau}(i) = i\} \subset \{\widetilde{\tau}(i) = i\}$ by (5). Hence, we obtain on the set $\{\bar{\tau}(i) = i\}$,

$$Y(i;\bar{\tau}) = Z(i) \ge \widetilde{Y}(i;\tau).$$

However, on $\{\bar\tau(i) > i\}$ the induction hypothesis yields,

$$\begin{aligned} Y(i;\bar\tau) &= E^{\mathcal{F}_i}\left[Z(\bar\tau(i+1))\right] = E^{\mathcal{F}_i}\left[Y(i+1;\bar\tau)\right] \geq E^{\mathcal{F}_i}\left[\widetilde{Y}(i+1;\tau)\right] \\ &= E^{\mathcal{F}_i}\left[\max_{i+1\leq p\leq k} E^{\mathcal{F}_{i+1}}\left[Z(\tau(p))\right]\right] \geq \max_{i+1\leq p\leq k} E^{\mathcal{F}_i}\left[Z(\tau(p))\right] \\ &= \widehat{Y}(i,\tau). \end{aligned}$$

Property (5) implies $\{\bar\tau(i) > i\} \subset \{\widehat\tau(i) > i\}$. Thus, on $\{\bar\tau(i) > i\}$,

$$\widehat{Y}(i,\tau) \geq Z(i)$$

and, by (6),

$$\widehat{Y}(i,\tau) = \widetilde{Y}(i,\tau) \quad \text{on } \{\bar\tau(i) > i\}.$$

This completes the proof of step 1. The second inequality now follows from (7) with the particular choice $\bar\tau = \widetilde\tau$.

Step 2: It remains to show that

$$Y(i;\bar\tau) \geq Y(i;\widetilde\tau).$$

For $i = k$ even equality holds. Suppose $0 \leq i \leq k-1$ and the inequality is proved for $i+1$. Then, on $\{\bar\tau(i) > i\} \cap \{\widetilde\tau(i) > i\}$,

$$Y(i,\bar\tau) = E^{\mathcal{F}_i}\left[Y(i+1,\bar\tau)\right] \geq E^{\mathcal{F}_i}\left[Y(i+1,\widetilde\tau)\right] = Y(i,\widetilde\tau)$$

by induction hypothesis. On $\{\bar\tau(i) > i\} \cap \{\widetilde\tau(i) = i\}$

$$Y(i,\bar\tau) \geq Z(i) = Y(i,\widetilde\tau)$$

by step 1. Finally, the set $\{\bar\tau(i) = i\} \cap \{\widetilde\tau(i) > i\}$ is evanescent by (5).

■

Suppose, for the time being, the sequence $\widetilde\tau^{(N)}(i)$ converges P-a.s. to some stopping time $\bar\tau(i)$. Clearly, $\bar\tau$ is, as a limit of consistent stopping families, itself a consistent stopping family. It can be shown by backward induction over i, that $\bar\tau$ satisfies (5). Indeed, the basic idea is as follows. Assume $\bar\tau(i) = i$. Then, for $N \geq N_0(\omega)$ sufficiently large

$$\widetilde\tau^{(N)}(i) = i,$$

i.e.

$$Z(i) \geq \max_{i+1\leq p\leq k} E^{\mathcal{F}_j}\left[Z(\tau(p))\right] + \epsilon^{(N)}(i).$$

We can now send N to infinity and obtain

$$Z(i) \geq \max_{i+1 \leq p \leq k} E^{\mathcal{F}_j}\left[Z(\tau(p))\right],$$

i.e.

$$\widetilde{\tau}(i) = i.$$

Thus, on $\{\bar{\tau}(i) = i\}$,

$$\widetilde{\tau}(i) \leq \bar{\tau}(i) \leq \widehat{\tau}(i).$$

A similar argument, making use of the induction hypothesis, yields the inequalities on $\{\bar{\tau}(i) > i\}$.

We can now define $\bar{\tau}^{(N)} = \bar{\tau}$ and (ia) is satisfied.

Unfortunately, example (page 352) shows that we may not expect $\widetilde{\tau}^{(N)}(i)$ to converge in general. Nonetheless, the previous considerations point to the right path. For ω such that $\widetilde{\tau}^{(M)}(i;\omega)$ converges, we define $\bar{\tau}^{(N)}(i;\omega)$ as this limit for all N. Otherwise, we define $\bar{\tau}^{(N)}(i;\omega) = i$, if and only if a subsequence of $\widetilde{\tau}^{(M)}(i;\omega)$ converges to i and $\widetilde{\tau}^{(N)}(i;\omega) = i$. The intuition is, that in the latter case we are free to choose the limit of any subsequence in order to obtain (5). So we choose $\bar{\tau}^{(N)}(i;\omega)$ as close as possible to $\widetilde{\tau}^{(N)}(i;\omega)$.

This reasoning can be formalized as follows. Define,

$$\bar{\tau}^{(N)}(k) = k$$

and

$$\begin{aligned}
\bar{\tau}^{(N)}(i) = i \quad &\iff \quad (\widetilde{\tau}^{(M)}(i) > i \text{ for only finitely many } M) \\
&\qquad \vee (\widetilde{\tau}^{(M)}(i) = i \text{ for infinitely many } M \text{ and } \widetilde{\tau}^{(N)}(i) = i) \\
\bar{\tau}^{(N)}(i) \neq i \quad &\implies \quad \bar{\tau}^{(N)}(i) = \bar{\tau}^{(N)}(i+1).
\end{aligned}$$

We have:

Lemma 3.3 $\bar{\tau}^{(N)}$ *satisfies (ia).*

The details of the proof can be found in Bender and Schoenmakers (2006), theorems 4.2 and 4.3. Stability of the algorithm, not only of one improvement step is also proven in this paper.

Remark 3.4 *Since* $\bar{\tau}^{(N)}(i) \leq \widehat{\tau}(i)$, *we obtain,*

$$\limsup_{N\to\infty} \widetilde{\tau}^{(N)}(i) \leq \widehat{\tau}(i).$$

On the other hand, the supermartingale property of the Snell envelope yields

$$\widehat{\tau}(i) \leq \widehat{\tau}^*(i) = \inf\left\{j:\ i \leq j \leq k,\ Z(j) > E^{\mathcal{F}_j}[Y^*(j+1)]\right\}.$$

As $\widehat{\tau}^*$ is the family of 'last optimal stopping times', we may conclude that the sub-optimality of $\widetilde{\tau}^{(N)}$ (for large N) basically stems from exercising to early.

4 Numerical examples

We now illustrate our algorithm with two examples: Bermudan basket-put and maximum-call options on 5 assets. We assume, that each asset is governed under the risk-neutral measure by the following SDE:

$$dS_i(t) = (r-\delta)S_i(t)dt + \sigma S_i(t)dW_i(t), \quad 1 \leq i \leq 5,$$

where $(W_1(t), \ldots, W_5(t))$ is a standard 5-dimensional Brownian motion. Suppose that an option can be exercised at $k+1$ dates $T_0, \ldots, T_k$, uniformly distributed between $T_0 = 0$ and $T_k = 3$(yr). The discounted price of the option is given by (1) with

$$Z(i) = e^{-rT_i}(K - \frac{S_1(T_i) + \cdots + S_5(T_i)}{5})^+ \quad \text{for the basket-put option and}$$
$$Z(i) = e^{-rT_i}(\max\{S_1(T_i), \ldots, S_5(T_i)\} - K)^+ \quad \text{for the maximum-call option.}$$

For our simulation, we take the following parameter values,

$$r = 0.05, \quad \sigma = 0.2, \quad S_1(0) = \ldots = S_5(0) = S_0, \quad K = 100,$$
$$\delta = 0 \text{ for basket-put option}, \quad \delta = 0.1 \text{ for maximum-call option.}$$

We consider options 'out-of-the-money', 'at-the-money' and 'in-the-money' at $t = 0$. For an initial stopping family $(\tau(i):\ 0 \leq i \leq k)$, we construct the lower bound $Y(0;\tau)$, an improved lower bound $Y(0;\widetilde{\tau})$ with $\widetilde{\tau}$ given by (3), and the dual upper bound $Y_{up}(0;\tau)$ given by (4). A natural 'intuitively good' initial exercise rule is to exercise, when the cashflow is larger than the maximal value of all still-alive European options:

$$\tau(i) = \inf\{j \geq i : Z(j) \geq \max_{p \geq j+1} E^{\mathcal{F}_j} Z(p)\},$$

which is in fact a one-step improvement of the trivial exercise policy $\tau(i) \equiv i$. For our examples, however, a closed-form expression for still-alive Europeans $E^{\mathcal{F}_j} Z(p)$, $p > j$

does not exist. Fortunately, a good closed-form approximation is available for the basket-put option. For the maximum-call option we improve upon the exercise rule, suggested by Andersen (1999), Strategy 1. We will show that in all examples our method gives Bermudan prices with a relative accuracy better than 1%.

4.1 Bermudan basket-put

In this example we approximate still-alive European options by a moment-matching procedure. Let us define $f(T_j) := (S_1(T_j) + \cdots + S_5(T_j))/5$ for $0 \le j \le k$ and take j, p with $j \le p \le k$. First, we approximate $f(T_p)$ by

$$f_j(T_p) := f(T_j) \exp\left((r_j - \frac{1}{2}\sigma_j^2)(T_p - T_j) + \sigma_j (W(T_p) - W(T_j)) \right),$$

where the parameters r_j and σ_j are taken in such a way that the first two moments of $f(T_p)$ and $f_j(T_p)$ are equal conditional $\mathcal{F}_j$:

$$\begin{aligned} r_j &= r, \\ \sigma_j &= \frac{1}{T_p - T_j} \ln \left(\frac{\sum_{m,n=1}^{5} S_m(T_j) S_n(T_j) \exp(1_{m=n} \sigma^2 (T_p - T_j))}{(\sum_{m=1}^{5} S_m(T_j))^2} \right), \end{aligned}$$

see, e.g., Brigo et al. (2004), Lord (2005). Then, we approximate $E^{\mathcal{F}_j} Z(p)$ by $E^{\mathcal{F}_j}[e^{-rT_p}(K - f_j(T_p))^+]$ using the Black-Scholes formula,

$$E^{\mathcal{F}_j}[e^{-rT_p}(K - f_j(T_p))^+] = e^{-rT_j} BS(f(T_j), r, \sigma_j, K, T_p - T_j),$$

and define the initial stopping family

$$\tau(i) := \{ j \le i : \ Z(j) \ge e^{-rT_j} \max_{j+1 \le p \le k} BS(f(T_j), r, \sigma_j, K, T_p - T_j) \}, \quad 0 \le i \le k.$$

Note that the initial stopping family $(\tau(i) : \ 0 \le i \le k)$ leads already to a reasonable lower approximation $Y(0;\tau)$ of the Bermudan price (less then 5% relative). The gap between the improved lower bound $Y(0;\tilde{\tau})$ and dual upper bound $Y_{up}(0;\tau)$ does not exceeds 1% relative. See table 1, where we used 10^7 Monte Carlo trajectories for $Y(0;\tau)$ and 2000 trajectories (with 1000 nested trajectories) for $Y_{up}(0;\tau)$. To simulate $Y(0;\tilde{\tau})$ we use 10^5 outer and 500 inner trajectories. An obvious variance reduction is obtained by simulating $Y_{up}(0;\tau) - Y(0;\tau)$ and $Y(0;\tilde{\tau}) - Y(0;\tau)$ rather than $Y_{up}(0;\tau)$ and $Y(0;\tilde{\tau})$.

k	S_0	$Y(0;\tau)$ (SD)	$Y(0;\widetilde{\tau})$ (SD)	$Y_{up}(0;\tau)$ (SD)
	90	10.000(0.000)	10.000(0.000)	10.000(0.002)
3	100	2.156(0.001)	2.158(0.002)	2.162(0.001)
	110	0.537(0.001)	0.537(0.001)	0.538(0.001)
	90	10.000(0.000)	10.000(0.000)	10.000(0.002)
6	100	2.361(0.001)	2.395(0.004)	2.406(0.003)
	110	0.571(0.001)	0.578(0.002)	0.578(0.001)
	90	10.000(0.000)	10.000(0.000)	10.001(0.002)
9	100	2.387(0.001)	2.471(0.005)	2.490(0.006)
	110	0.579(0.001)	0.594(0.002)	0.596(0.002)

Table 1: Bermudan basket-put on 5 assets

4.2 Bermudan maximum-call

In contrast to the previous example, no good approximations are known for the still-alive maximum-call Europeans. For this example we take as initial stopping family strategy I of the Andersen method (see Andersen (1999)):

$$\tau(i) = \inf\{j \geq i : Z(j) \geq H_j\}.$$

The sequence of constants H_j is pre-computed using $5 \cdot 10^5$ simulations. Note that the gap between Andersen's lower bound $Y(0;\tau)$ and its dual upper bound $Y_{up}(0;\tau)$ varies from 2% to 4%, see columns 3 and 5 in table 2 (we use $5 \cdot 10^6$ Monte Carlo trajectories for $Y(0;\tau)$ and 500 Monte Carlo trajectories (with 1000 inner simulations for $Y_{up}(0;\tau) - Y(0;\tau)$). Further, we construct the improvement $Y(0;\widetilde{\tau})$ of Andersen's lower bound using 10^4 outer and 1000 inner simulations. The results are compared with the 90% confidence interval of Broadie and Glasserman (2004) computed by the stochastic mesh method, see table 2. We see that in almost all cases, $Y(0;\widetilde{\tau})$ and $Y_{up}(0;\tau)$ is within the 90% confidence interval, and that the gap between them does not exceed 1%.

Remark 4.1 *The cross-sectional least square algorithm by Longstaff and Schwartz (2001) yields results consistent with B-G: The lower bound reported in Longstaff and Schwartz (2001) for $d = 9$ and 19 basis functions are 16.657, 26.182, and 36.812, respectively. Slightly lower values are reported in Andersen and Broadie (2004) with 13 basis functions.*

k	S_0	$Y(0;\tau)$ (SD)	$Y(0;\widetilde{\tau})$ (SD)	$Y_{up}(0;\tau)$ (SD)	90% Confidence interval by BG
	90	15.702(0.008)	16.026(0.033)	15.986(0.021)	[15.995, 16.016]
3	100	24.716(0.009)	25.244(0.044)	25.333(0.031)	[25.267, 25.302]
	110	34.856(0.011)	35.695(0.056)	35.745(0.037)	[35.679, 35.710]
	90	16.064(0.007)	16.394(0.080)	16.462(0.054)	[16.438, 16.505]
6	100	25.171(0.009)	25.751(0.107)	25.978(0.066)	[25.889, 25.948]
	110	35.399(0.010)	36.329(0.131)	36.523(0.079)	[36.466, 36.527]
	90	16.202(0.007)	16.681(0.079)	16.734(0.063)	[16.602, 16.710]
9	100	25.343(0.009)	26.118(0.110)	26.333(0.083)	[26.101, 26.211]
	110	35.605(0.010)	36.652(0.134)	37.028(0.100)	[36.719, 36.842]

Table 2: Bermudan maximum-call on 5 assets

Concluding remarks

The iterative Monte Carlo procedures for pricing callable structures reviewed in this paper are quite generic as in principle it only requires a Monte Carlo simulation mechanism for an underlying Markovian system, for instance a Markovian system of SDEs. Moreover, by incorporating information obtained from another suboptimal method, for example Andersen's method (see Andersen (1999)) or the method of Longstaff and Schwartz (2001), we may improve upon this method to obtain our target results more efficiently.

The iterative procedures can be easily adapted to a large class of path-dependent exotic instruments where a call generates a sequence of cash-flows in the future. For these products one may construct 'virtual cash-flows' which are basically present values of future cash-flows specified in the contract. An important example is the (cancellable) snowball swap, an exotic interest rate product with growing popularity. In Bender et al. (September 2006) this product is treated in the context of a full-blown Libor market model (structured as in Schoenmakers (2005)). From this treatment it will be clear how to design Monte Carlo algorithms for related callable path-dependent Libor products.

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